Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

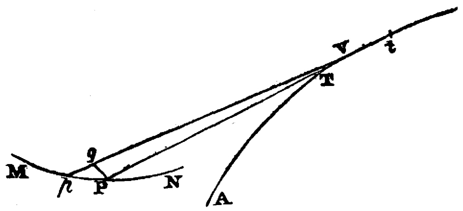
JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Geometrical Theorems on the Rectification of the Conic Sections.

By JAMES MAC CULLAGH, ESQ. *Communicated by the* REV. DR. SADLEIR, S. F. T. C. D., *Vice President.*

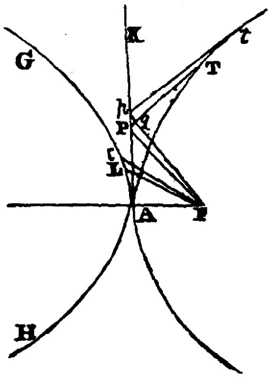
Read June 21st, 1830.

Lemma I. Let T and t be two points indefinitely near each other on any given curve AT, and let tangents at T and t meet in the points P and p any other given line MN, straight or curved, and draw Pq perpendicular to tp; then the difference between the arc AT and the tangent TP will exceed or fall short of the difference between the arc At and the tangent tp by a quantity which is ultimately to pq in a ratio of equality.



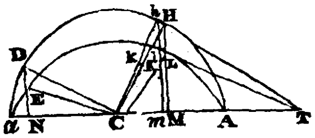
For the increment of TP, or the difference of TP and tp, is ultimately equal to the sum of pq, VT, and Vt, (V being the intersection of pt and PT produced); and Tt, or the increment of the arc AT, is ultimately equal to the sum of VT and Vt; therefore the difference of the increments is ultimately equal to pq. Whence the proposition is manifest.

PROB. To find the Length of the Arc of a Parabola.



Let F be the focus and A the vertex of a parabola AT: draw AK perpendicular to AF, and let a tangent at the point T meet it in P; take *p* indefinitely near to P, and let F*p* intersect PT in *q*: then since FP and F*p* are perpendicular to the tangents PT and *pt*, it follows (by the preceding lemma) that P*q* is ultimately equal to the increment of the difference between the arc AT and the tangent TP. With the centre F and semiaxis FA describe the equilateral hyperbola GAH, and bisect the angles AFP, AF*p* by the straight lines FL, F*l*. Then (since the square of the radius vector of an equilateral hyberbola is inversely as the cosine of twice the angle which it makes with the axis) the square of LF will be equal to AF × PF; and because the angle LF*l* is half the angle PF*q*, therefore the area LF*l* will be equal to one-fourth of the rectangle under AF and P*q*.

Hence the hyperbolic sector AFL is equal to one-fourth of the rectangle under AF and the difference between the parabolic arc AT and the tangent TP.



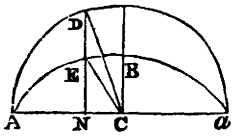
Lemma 2. If on either axis of an ellipse a semicircle be described, of which CD and CH are two radii at right angles to each other, and if DN and HM be drawn perpendicular to the axis *a*A, and meeting the ellipse in E and L; then CE and CL will be conjugate semidiameters.

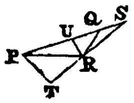
For tangents at H and L will meet in a point T in A*a* produced; the triangles HMT and DNC will be similar, and DN and HM will be similarly divided in E and L, therefore CE and TL will be parallel, and consequently CE and CL will be conjugate semidiameters.

Lemma 3. Take in the ellipse a point *l* indefinitely near to L, and draw through it *mh* parallel to MH; join C*h*, and with the centre C and a radius equal to CE describe the arc K*k* meeting CH and C*h* in K and *k*: then K*k* will be ultimately equal to L*l*.

For L*l* : H*h* :: LT : HT :: CE : CD :: CK : CH :: K*k* : H*h*. Therefore L*l* = K*k*.

Lemma 4. If the semiaxes AC and BC of an ellipse be equal to the sum and difference of the sides PQ, QR, of a triangle PQR, and if the angle BCD be equal to half the contained angle PQR (AD*a* being the semicircle on A*a*); then, DEN being drawn perpendicular to A*a*, CE will be equal to the base PR. Take QU and QS equal to QR; then PS and PU will be equal to AC and CB, and the angle CDN to TSP; therefore drawing PT perpendicular to TS, DN and NC will be equal to TS and TP. But URS being a right angle, UR is parallel to PT, and therefore TS : TR :: PS : PU :: AC : BC :: DN : EN; but TS = DN, therefore TR = EN, and since PT = CN, it follows that PR = CE.

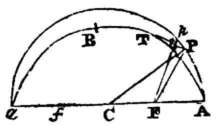


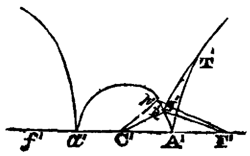


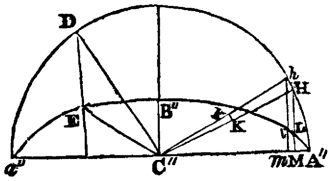
THEOREM.

Let AT and AT′ be an ellipse and hyperbola, the semiaxis (CA or C′A′) of either being equal to (C′F′ or CF) the distance between the focus and centre of the other; and let tangents at the points T and T′ meet in P and P the circles described on the axes, so that FP = F′P′:

let also *a*″ B″ A″ be another ellipse whose semiaxes (A″ C″ and B″ C″) are equal to *a*F and FA, and take in its circumference a point L so that the semidiameter conjugate to that passing through L may be equal to FP or F′P′; then will the excess of the elliptic arc AT above its tangent TP be greater than the excess of the hyperbolic arc A′T′ above its tangent T′P′, by twice the elliptic arc A″L.







Take the point L so that when the ordinate MLH is drawn to meet in H the semicircle described on the axis, the angle HC″M may be equal to half the angle PCF (or P′C′F′, for the triangles PCF and P′C′F′ have all their sides and angles equal); draw C″D perpendicular to C″H and DE to A″a″ meeting the ellipse in E; then by lem. 2, C″E will be conjugate to C′L, and by lem. 4, it will be equal to FP, since the angle B″C″D is equal to HC″M and is therefore half of PCF, whilst the semiaxes C″A″, C″B″, are the sum and difference of PC and CF. Hence the point L thus found is that required by the enunciation. Take *p*, *p*′, *h*, indefinitely near to P, P′, H, and similarly related to each other; let F*p* and F*p*′ intersect TP and T′P in *q* and *q*′, and with the centre C′ and a radius equal to C″E describe the evanescent arc K*k*. Then FP, F′P′ are always perpendidular to TP, TP′, and therefore P*q* is ultimately the increment of the difference between the arc AT and the tangent TP (*lem.* 1.) and P′*q*′ the increment of the difference between the arc A′T′ and the tangent TP′; also by *lem.* 3. K*k* is ultimately equal to L*l* the increment of the arc A″L. Now the angles CFP and CF*p* are equal to C′P′F′ and C′*p*′F′; therefore $\mathrm{PF}p$ is equal to the sum of $\mathrm{P'F'}p'$ and $\mathrm{P'C'}p'$, or to $\mathrm{P'F'}p'$ with twice $\mathrm{HC''}h$: but FP, F'P', and C''K are all equal, and therefore $\mathrm{P}q$ is equal to $\mathrm{P'}q'$ and twice $\mathrm{K}k$, or to $\mathrm{P'}q'$ and twice $\mathrm{L}l$. Whence the proposition is manifest.

Schol. The angle B''C''H is half the angle $f\mathrm{CP}$ or $f'\mathrm{C'P'}$, and therefore by *lem.* 4, the semidiameter C''L is equal to the straight line $f\mathrm{P}$ or $f'\mathrm{P'}$. This gives another and easier method of finding the point L.

Hence every arc of a hyperbola may be found by means of two elliptic arcs. This beautiful theorem was discovered by Landen, and is now for the first time demonstrated geometrically; but the manner in which it is stated by him and by succeeding writers differs from the above, and is much more complicated. The different forms under which they have presented it may be easily deduced from the preceding, by means of the theorem of Fagnani, of which a geometrical demonstration was first given in Vol. IX. of these Transactions, by the Lord Bishop of Cloyne.

December 23, 1829.